PIONEER
ACADEMIC PUBLISHING LIMITED

Journal of Progress in Engineering and Physical Science
ISSN 2709-4006
www.pioneerpublisher.com/jpeps
Volume 2 Number 2 June 2023

Singular Optimal Linear Quadratic Regulator of Switched Systems

Lulu Fu[1]

[1] School of Liberal Studies, Liaoning University of International Business and Economics, Dalian, 116052, China
Correspondence: Lulu Fu, School of Liberal Studies, Liaoning University of International Business and Economics, Dalian, 116052, China.

doi:10.56397/JPEPS.2023.06.04

Abstract

In this paper, we will investigate the singular linear quadratic regulator (LQR) problem of switched linear system in finite time horizon. The proposed method is transforming into a switched LQR problem by adopting linear transformation. Next, we adopt an embedding transformation method to convert the switched LQR problem to a traditional optimal control problem, so the bang-bang-type solution of the embedded optimal control problem in finite time is the optimal solution to the switched LQR problem. The switching sequence of modes and the switching instants can be calculated by solving a closed-form optimal switching condition. The optimal state feedback control law is determined simultaneously. Then, by solving a sequence of Riccati equation, we find some conditions that ensure switched LQR problem can be convert to the singular LQR problem. Finally, a numerical example is presented to demonstrate the effectiveness of the proposed method.

Keywords: Switched systems, Linear transformation, Embedding transformation, Quadratic programming, Riccati equation

1. Introduction

As an important class of hybrid system, switched systems have drawn considerable attention in the past thirty years (Liu Xiaomeng, et al., 2013; Lu Junjie et al., 2018; Fu Jun et al., 2015; Lee Ti-Chung et al., 2017; Xu Wei et al., 2020; Chen Weisheng et al., 2018). Switched systems have wide range of practical engineering application, such as aerospace field, chemical, biology and economics. In addition, different properties, such as stability (Ma Ruicheng & An Shuang, 2019), stabilization (Ma Ruicheng, Chen Qi, Zhao Shengzhi & Fu Jun, 2021), controllability (Liu Xiaomeng, Lin Hai & Chen Ben M., 2013), observer design (Tanwani Aneel, Shim Hyungbo & Liberzon Daniel, 2013), and H_∞ control (Ma Ruicheng, Ma Mingjun, Li Jinghan, Fu Jun & Wu Caiyun, 2019), of switched systems are one of the hot topics in the literature. Some effective research methods, for example, common Lyapunov function (Ma Ruicheng, Liu Yan, Zhao Shengzhi, Wang Min & Zong Guangdeng, 2015), single Lyapunov function (Wang Min & Zhao Jun, 2010), and multiple Lyapunov functions (Li Li Li, Zhao Jun & Dimirovski Georgi M., 2013), play an important role in investigating switched systems.

The study of optimal control is an important research content of modern control theory (Niu Teng, et al., 2018; Sorin C. Bengea et al., 2005; Riedinger & Pierre, 2014; Xu, Wei et al., 2020; Xu Wei, et al., 2017). The basic methods of studying optimal control mainly include three methods: variational method, minimum value principle and dynamic programming (Luus Rein, & Chen Yang Quan, 2004; Seatzu Carla, Corona Daniele, Giua Alessandro & Bemporad Alberto, 2006; Xu Xuping, Antsaklis Panos J., 2004). In recent years, the optimal control of switched systems has attracted increasing attention because of their importance from both theoretical and practical points of view. In order to achieve the optimal control of a switched system, one needs to determine a subsystem sequence, fix the switching times between the subsystems and design an input for each subsystem. It should be noted that they are strongly coupled. Therefore, the optimal control of switched systems is much more difficult than the one of non-switched systems. Some classical approaches, such as minimum value principle and dynamic programming, have extended to investigate the optimal control problem of switched systems. Since the linear quadratic regulation (LQR) problem (Duarte J. Antunes & W.P.M.H. Heemels, 2017; Seatzu Carla, Corona Daniele, Giua Alessandro & Bemporad Alberto, 2006; Bijl Hildo & Schon Thomas B., 2019; Wu Weiping, Gao Jianjun, Lu Jun Guo & Li Xun, 2020) is very commonly used in optimal control applications, we consider the global optimal solution of this class of optimal switching problem in this paper. As a special class of LQR problems, a basic problem of a switched system is to find an optimal switching times with a fixed predefined mode sequence such that the objective function is optimal. Although various method has also been developed to deal with LQR problem for various classes of switched systems, the question of how to obtain a closed-form optimal solution of the switching sequence and the control input is still a challenging problem. Applying the embedding-transformation method, (Wu Guangyu, Sun Jian & Chen, Jie, 2019) investigate two closed-form switching conditions involved by the switching law for LQ cost and multiple LQ cost when the mode sequence and the switching instants are unspecified. The switching-dependent state feedback control law can be determined simultaneously. Since there exist the control input in the objective function, which makes the Hamilton function and the control variable have a nonlinear relationship. However, when the control input does not exist in the objective function, the Hamiltonian has a linear relationship with the control variable, which will yield the singular LQR problem of switched systems. Although some efforts have been done for the LQR problem of switched systems, there are few results on the singular LQR of switched systems in the literature.

In this paper, we will investigate the singular LQR problem of switched linear system in finite time horizon. First, a linear transformation is introduced, which converts the singular LQR problem into the switched LQR problem. Second, the embedding transformation method is then adopted to convert the switched LQR problem to the continuous optimal control problem. The optimal switch input can be viewed as a quadratic programming problem. The quadratic programming problem is considered as a minimization of a concave function. The optimal solution of the switched LQR problem is of bang- bang type. Then, both the control input and the switching signal are simultaneously designed. Next, by solving a sequence of Riccati equation, some conditions are shown to ensure that the switched LQR problem can be convert to the singular LQR problem. Therefore, both the closed-loop system of the singular LQR problem and optimal switching condition of subsystems can be obtained. Finally, a numerical example is presented to demonstrate the effectiveness of the proposed method.

The paper organization is as follows. In Section 2, the problem statements and preliminaries are presented. Main results are given in Section 3. A numerical example is shown to illustrate the validity of the theoretical results in Section 4. Finally, some conclusions are drawn in Section 5.

2. Problem Statements and Preliminaries

In this paper, we consider the following class of switched linear system:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + Bu(t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^{n1}$ is the control input, $\sigma(t):[0,\infty) \to M = \{1,2,\cdots,m\}$ is the switching law which is assumed to be a piecewise continuous (from the right) function of time, with m being the number of subsystems, A_p and B, $\forall p \in M$, are known matrices of the appropriate dimensions, t_0 is a fixed initial time and $x(t_0)$ is the initial state.

Now, we make the following assumption for switched system (1).

Assumption 1: Each subsystem $(A_i, B), \forall i \in M$, is controllable.

In this paper, we will study the singular linear quadratic regulation (SLQR) problem:

Problem 1: For the SLQR problem of switched system (1), the control input $u(t)$ and switching signal $\sigma(t)$ will be co-designed to minimize the following cost function:

$$J = \frac{1}{2}\int_{t_0}^{t_f} x^T(t)Qx(t)dt, \tag{2}$$

where Q is a $n \times n$ positive semi-definite matrix, and t_f is a fixed final time.

In the following, we need to explain why Problem 1 is singular.

First construct the Hamiltonian function:

$$H = \frac{1}{2}x^T(t)Qx(t) + \lambda(t)[A_{\sigma(t)}x(t) + Bu(t)], \tag{3}$$

where $\lambda(t)$ is the Lagrangian multiplier. At this time, the Hamiltonian function H of Problem 1 has a linear relationship with the control variable $u(t)$ and a nonlinear relationship with the state variable $x(t)$.

According to the minimum value principle, $x(t)$ and $\lambda(t)$ satisfy the following regular equations:

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda} = A_{\sigma(t)}x(t) + Bu(t), \tag{4}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -Qx(t) - A_{\sigma(t)}^T\lambda(t). \tag{5}$$

Extreme value conditions:

$$\frac{\partial H}{\partial u} = B^T\lambda(t) = 0, \tag{6}$$

$$\frac{\partial^2 H}{\partial u^2} = 0. \tag{7}$$

For non-zero $\lambda(t)$, the optimal control is

$$u(t) = -\operatorname{sgn}\{B^T\lambda(t)\}. \tag{8}$$

(8) shows that the optimal control takes a value on its constraint boundary, which is a bang-bang control form.

Although the extreme value condition satisfies the necessary condition of the minimum value principle, the Hamiltonian function has nothing to do with the control, so the Hamiltonian function cannot be the absolute minimum relative to $u(t)$. This kind of control problem is called singular optimal control problem.

In order to use the results of the standard regulator to obtain the solution of Problem 1, the singular regulator can be transformed into an equivalent standard regulator by the linear transformation method, that is, the modified singular linear quadratic regulator (MSLQR).

First, in order to solve Problem 1, we define the following linear transformation on the switched system (1):

$$x_1(t) = x(t) - Bu_1(t), \tag{9}$$

$$\dot{u}_1(t) = u(t). \tag{10}$$

From (9) and (10), we get

$$\begin{aligned}\dot{x}_1(t) &= \dot{x}(t) - Bu(t)\\ &= A_{\sigma(t)}x(t) + Bu(t) - Bu(t)\\ &= A_{\sigma(t)}x(t) - A_{\sigma(t)}Bu_1(t) + A_{\sigma(t)}Bu_1(t)\\ &= A_{\sigma(t)}[x(t) - Bu_1(t)] + A_{\sigma(t)}Bu_1(t)\\ &= A_{\sigma(t)}x_1(t) + \tilde{B}_{\sigma(t)}u_1(t),\end{aligned} \tag{11}$$

where

$$\tilde{B}_{\sigma(t)} = A_{\sigma(t)}B. \tag{12}$$

Define

$$H = QB, \tag{13}$$

$$R = B^T QB. \tag{14}$$

One can check that R is a symmetric positive definite matrix. Then, substituting (9) into (2) yields that

$$\begin{aligned}J &= \frac{1}{2}\int_{t_0}^{t_f} x^T(t)Qx(t)dt\\ &= \frac{1}{2}\int_{t_0}^{t_f} [x_1(t)+Bu_1(t)]^T Q[x_1(t)+Bu_1(t)]dt\\ &= \frac{1}{2}\int_{t_0}^{t_f} [x_1^T(t)Qx_1(t)+x_1^T(t)Hu_1(t)+u_1^T(t)H^T x_1(t)+u_1^T(t)Ru_1(t)]dt\\ &= \frac{1}{2}\int_{t_0}^{t_f} [x_1^T(t)Qx_1(t)-x_1^T(t)HR^{-1}H^T x_1(t)+x_1^T(t)HR^{-1}Ru_1(t)+u_1^T(t)RR^{-1}H^T x_1(t)+x_1^T(t)HR^{-1}RR^{-1}H^T x_1(t)+u_1^T(t)Ru_1(t)]dt\\ &= \frac{1}{2}\int_{t_0}^{t_f} [x_1^T(t)Q_1x_1(t)+u_2^T(t)Ru_2(t)]dt,\end{aligned} \tag{15}$$

where

$$Q_1 = Q - HR^{-1}H^T, \tag{16}$$

$$u_2(t) = u_1(t) + R^{-1}H^T x_1(t). \tag{17}$$

As a performance index, in (15), Q_1 and R are required to be symmetric non-negative definite and positive definite matrices. As for the non-negative qualitativeness of Q_1, the following proposition can be seen.

Proposition 1: If $Q \geq 0, R > 0$, then $Q_1 = Q - HR^{-1}H^T \geq 0.$

Proof: Due to $Q \geq 0$, one has

$$x^T(t)Qx(t) = x_1^T(t)Q_1x_1(t) + [u_1(t)+R^{-1}H^T x_1(t)]^T R[u_1(t)+R^{-1}H^T x_1(t)] \geq 0, \forall x(t). \tag{18}$$

Define $u_1(t) = -R^{-1}H^T x_1(t)$. Then, we obtain that

$$x_1^T(t)Q_1x_1(t) \geq 0, \forall x_1(t).$$

Therefore, we have $Q_1 \geq 0$.

Remark 1: If $Q > 0$ and $rankB = m$, there must be $R > 0$ in (14). For the case of $Q \geq 0$, there are many possible matrices B, which can make $R > 0$. If R is not positive definiteness, we transformed it until the positive definiteness is established. In the following, we assume that $R > 0$.

It should be noted that (15) does not contain $u_1(t)$ but contains $u_2(t)$. Thus, (11) can be further transformed.

Substituting (17) into (11) yields

$$\begin{aligned}\dot{x}_1(t) &= A_{\sigma(t)}x_1(t)+\tilde{B}_{\sigma(t)}u_1(t)\\ &= A_{\sigma(t)}x_1(t)+\tilde{B}_{\sigma(t)}[u_2(t)-R^{-1}H^Tx_1(t)]\\ &= A_{\sigma(t)}x_1(t)+\tilde{B}_{\sigma(t)}u_2(t)-\tilde{B}_{\sigma(t)}R^{-1}H^Tx_1(t)\\ &=[A_{\sigma(t)}-\tilde{B}_{\sigma(t)}R^{-1}H^T]x_1(t)+\tilde{B}_{\sigma(t)}u_2(t)\\ &=\tilde{A}_{\sigma(t)}x_1(t)+\tilde{B}_{\sigma(t)}u_2(t)\end{aligned} \tag{19}$$

and $x_1(t_0)=x(t_0)-Bu_1(t_0)$, where

$$\tilde{A}_{\sigma(t)}=A_{\sigma(t)}-\tilde{B}_{\sigma(t)}R^{-1}H^T. \tag{20}$$

In order to ensure the existence of the optimal solution of (19), we assume that each subsystem $(\tilde{A}_i,\tilde{B}_i),\forall i\in M$, is controllable.

By using the linear transformation method, a new optimal control problem (MSLQR) can be defined as follows.

Problem 2: For switched system (19), the MSLQR problem can be defined as determining a control input $u_2(t)$ and a switch signal $\sigma(t)$ associated with a general LQ cost function for evaluating the systems performance quantitatively in a finite horizon $[t_0,t_f]$:

$$\min \quad J=\frac{1}{2}\int_{t_0}^{t_f}[x_1^T(t)Q_1x_1(t)+u_2^T(t)Ru_2(t)]dt, \tag{21}$$

where Q_1 and R are symmetric non-negative definite and positive definite matrices.

Remark 2: If B is reversible, then $\tilde{A}_{\sigma(t)}=0$ and $Q_1=0$.

3. Main Results

In this section, we first propose the main result for Problem 2.

Theorem 1: Consider the switched system (19), both the switching signal

$$\sigma(t)=\arg\min_{i\in M}\lambda^T(t)[\tilde{A}_ix_1(t)-\frac{1}{2}\bar{B}_{ii}\lambda(t)], \tag{22}$$

and the switched controller

$$u_2(t)=-R^{-1}\tilde{B}^T_{\sigma(t)}\lambda(t), \tag{23}$$

minimize the cost functional (21), where $\lambda(t)=[\lambda_1,\cdots,\lambda_n]^T$ is the solution of

$$\dot{\lambda}(t)=-Q_1x_1(t)-\tilde{A}^T_{\sigma(t)}\lambda(t), \tag{24}$$

with the boundary condition $\lambda(t_f)=0$.

Proof: For simplicity, we define

$$\sum_{i=1}^N w_i(t)\triangleq\sum_i w_i,\quad \sum_{i=1}^N\sum_{j=1}^N w_i(t)w_j(t)\triangleq\sum_{i,j}w_iw_j.$$

Then, switched system (19) can be represented by a combination of N subsystems:

$$\dot{x}_1(t)=\sum_i w_i(t)[\tilde{A}_ix_1(t)+\tilde{B}_iu_2(t)], \tag{25}$$

where $w_i(t)\in\{0,1\}$.

By adopting the embedding transformation method, we embed switched system (25) into a larger family of systems by allowing $w_i(t)$ to vary continuously in $[0,1]$.

The Problem 2 can be transformed into the embedded Problem 2 as follows:

$$\min \quad J=\frac{1}{2}\int_{t_0}^{t_f}[x_1^T(t)Q_1x_1(t)+u_2^T(t)Ru_2(t)]dt$$
$$s.t. \quad \dot{x}_1(t)=\sum_i w_i(t)[\tilde{A}_i x_1(t)+\tilde{B}_i u_2(t)]. \tag{26}$$

The time-varying vector $w(t)$ belongs to a convex set W:

$$W=\{w\in R^N:\sum_i w_i=1, w_i\geq 0\}. \tag{27}$$

The Hamilton function is defined as

$$H[x_1,u_2,w,\lambda]=\frac{1}{2}[x_1^T(t)Q_1x_1(t)+u_2^T(t)Ru_2(t)]+\lambda^T(t)\sum_i w_i(t)[\tilde{A}_i x_1(t)+\tilde{B}_i u_2(t)]. \tag{28}$$

Then, we obtain the adjoint equation and boundary conditions:

$$\dot{\lambda}(t)=-\frac{\partial H}{\partial x_1}=-Q_1x_1(t)-\sum_i w_i\tilde{A}_i^T\lambda(t), \tag{29}$$

$$\lambda(t_f)=0. \tag{30}$$

Since $u_2(t)$ is not constrained, then the optimal control should satisfy the following:

$$\frac{\partial H}{\partial u_2}=Ru_2(t)+\sum_i w_i\tilde{B}_i^T\lambda(t)=0. \tag{31}$$

Sine $R>0$, then its inverse R^{-1} exists. Thus, we have

$$u_2(t)=-R^{-1}\sum_i w_i\tilde{B}_i^T\lambda(t). \tag{32}$$

Substituting (32) into (28) yields

$$H[x_1,w,\lambda]=\frac{1}{2}x_1^T(t)Q_1x_1(t)+\lambda^T(t)\sum_i w_i\tilde{A}_i x_1(t)-\frac{1}{2}\lambda^T(t)\sum_{i,j}w_iw_j\tilde{B}_iR^{-1}\tilde{B}_j^T\lambda(t). \tag{33}$$

Minimizing H with respect to $w(t)$ can be simplified to minimize

$$\bar{H}[x_1,w,\lambda]=-\frac{1}{2}\lambda^T(t)\sum_{i,j}w_iw_j\tilde{B}_iR^{-1}\tilde{B}_j^T\lambda(t)+\lambda^T(t)\sum_i w_i\tilde{A}_i x_1(t). \tag{34}$$

Define

$$\bar{B}_{ij}\triangleq\tilde{B}_iR^{-1}\tilde{B}_j^T, \tag{35}$$

where $\tilde{B}_i=[\tilde{b}_1^i,\cdots,\tilde{b}_n^i]^T$ and $\tilde{b}_j^i(j=1,\cdots,n)$ is a m-dimensional row vector. The elements of $\bar{B}_{ij}$ can be obtained

$$\bar{B}_{ij}(s,t)=\tilde{b}_s^iR^{-1}(\tilde{b}_t^i)^T, \tag{36}$$

where $s,t=1,\cdots,n$.

Following the method in (Wu Guangyu, Sun Jian & Chen Jie, 2019), to minimize $\bar{H}$ with respect to $w(t)$ can be viewed as a quadratic programming problem:

$$\begin{aligned} &\min \quad -\frac{1}{2} w^T(t)G(t)w(t) + q^T(t)w(t) \\ &s.t. \quad w(t) \in W, \end{aligned} \tag{37}$$

Where $q(t) = [q_1, \cdots, q_N]^T$ and

$$G(i,j) = \lambda^T(t)\sum_{i,j} w_i w_j \overline{B}_{ij}(s,t)\lambda(t), \tag{38}$$

$$q_i = \lambda^T(t)\tilde{A}_i x_1(t). \tag{39}$$

One can check that

$$\overline{B}_{ij} = \tilde{B}_j R^{-1}\tilde{B}_i^T = \overline{B}_{ij}^T, \tag{40}$$

$$G(j,i) = \lambda^T(t)\overline{B}_{ji}\lambda(t) = G(i,j). \tag{41}$$

Then, matrix $G(t)$ is symmetric and $-G(t) \le 0$. Therefore, problem (37) is considered as a minimization of a concave function. In this case, the global minimum point of $\overline{H}$ is always achieved at the extreme point of the convex set W, i.e., the optimal solution of the embedded Problem 2 is of bang-bang type.

Therefore,

$$\begin{aligned} \overline{H}_m &= \min \overline{H} \\ &= \min_{i \in M} \lambda^T(t)[\tilde{A}_i x_1(t) - \frac{1}{2}\overline{B}_{ii}\lambda(t)] \\ &= \lambda^T(t)[\tilde{A}_k x_1(t) - \frac{1}{2}\overline{B}_{kk}\lambda(t)], \end{aligned} \tag{42}$$

where $w_k = 1$ and $w_i = 0, \forall i \ne k$. This completes the proof.

In the following, we will apply the solution of Problem 2 to solve Problem 1.

First, from (19) and (23), we get

$$\dot{x}_1(t) = \tilde{A}_{\sigma(t)}x_1(t) - \tilde{B}_{\sigma(t)}R^{-1}\tilde{B}_{\sigma(t)}^T\lambda(t). \tag{43}$$

It is clear that (24) and (43) are linear, i.e., $\lambda(t)$ and $x_1(t)$ are linear. Therefore, we can define that

$$\lambda(t) \triangleq P_{\sigma(t)}x_1(t), \tag{44}$$

where $P_{\sigma(t)}$ is the non-negative definite symmetric matrix. Then, (43) becomes

$$\dot{x}_1(t) = \tilde{A}_{\sigma(t)}x_1(t) - \tilde{B}_{\sigma(t)}R^{-1}\tilde{B}_{\sigma(t)}^T P_{\sigma(t)}x_1(t). \tag{45}$$

Taking the derivative of (44) with respect to time t, we have

$$\dot{\lambda}(t) = P_{\sigma(t)}\dot{x}_1(t), \tag{46}$$

together with (45), we obtain that

$$\dot{\lambda}(t) = [P_{\sigma(t)}\tilde{A}_{\sigma(t)} - P_{\sigma(t)}\tilde{B}_{\sigma(t)}R^{-1}\tilde{B}_{\sigma(t)}^T P_{\sigma(t)}]x_1(t). \tag{47}$$

Applying (44) to (24), it has

$$\dot{\lambda}(t) = -[Q_1 + \tilde{A}_{\sigma(t)}^T P_{\sigma(t)}] x_1(t). \tag{48}$$

According to (47) and (48), $P_{\sigma(t)}$ satisfies the following algebra Riccati equation:

$$0 = P_{\sigma(t)} \tilde{A}_{\sigma(t)} + \tilde{A}_{\sigma(t)}^T P_{\sigma(t)} - P_{\sigma(t)} \tilde{B}_{\sigma(t)} R^{-1} \tilde{B}_{\sigma(t)}^T P_{\sigma(t)} + Q_1. \tag{49}$$

Substituting (44) into (23), the optimal controller of the Problem 2 is

$$\begin{aligned} u_2(t) &= -R^{-1} \tilde{B}_{\sigma(t)}^T P_{\sigma(t)} x_1(t) \\ &\triangleq -K x_1(t), \end{aligned} \tag{50}$$

where

$$K = R^{-1} \tilde{B}_{\sigma(t)}^T P_{\sigma(t)}. \tag{51}$$

Then, substituting (50) into (17) yields

$$\begin{aligned} u_1(t) &= u_2(t) - R^{-1} H^T x_1(t) \\ &= -R^{-1} \tilde{B}_{\sigma(t)}^T P_{\sigma(t)} x_1(t) - R^{-1} H^T x_1(t) \\ &= -R^{-1} (\tilde{B}_{\sigma(t)}^T P_{\sigma(t)} + H^T) x_1(t) \\ &\triangleq -K_1 x_1(t), \end{aligned} \tag{52}$$

where

$$K_1 = -R^{-1} (\tilde{B}_{\sigma(t)}^T P_{\sigma(t)} + H^T) = (B^T Q B)^{-1} [B^T (\tilde{A}_{\sigma(t)}^T P_{\sigma(t)} + Q)]. \tag{53}$$

Therefore, the above results can be summarized as the following theorem.

Theorem 2: For Problem 2, its optimal control is shown in (52), where K_1 satisfies (53), $P_{\sigma(t)}$ satisfies (49) and $\sigma(t)$ satisfies (22). Each variable satisfies the following relationship (for all t):

$$P_{\sigma(t)} B = 0; \tag{54}$$

$$K_1 B = I; \tag{55}$$

$$K_1 x(t) = 0. \tag{56}$$

Proof: (i) Applying (20) and (16) to (49), one has

$$\begin{aligned} 0 = &P_{\sigma(t)} [A_{\sigma(t)} - \tilde{B}_{\sigma(t)} R^{-1} H^T] + [A_{\sigma(t)} - \tilde{B}_{\sigma(t)} R^{-1} H^T]^T P_{\sigma(t)} \\ &- P_{\sigma(t)} \tilde{B}_{\sigma(t)} R^{-1} \tilde{B}_{\sigma(t)}^T P_{\sigma(t)} + Q - H R^{-1} H^T. \end{aligned} \tag{57}$$

Multiplying matrix B to the right of (57), we can get that

$$\begin{aligned} 0 = &P_{\sigma(t)} [A_{\sigma(t)} - \tilde{B}_{\sigma(t)} R^{-1} H^T] B + [A_{\sigma(t)} - \tilde{B}_{\sigma(t)} R^{-1} H^T]^T P_{\sigma(t)} B \\ &- P_{\sigma(t)} \tilde{B}_{\sigma(t)} R^{-1} \tilde{B}_{\sigma(t)}^T P_{\sigma(t)} B + Q B - H R^{-1} H^T B. \end{aligned} \tag{58}$$

Substituting (12), (13), and (14) into (58), we have

$$0 = [A_{\sigma(t)}^T - H R^{-1} \tilde{B}_{\sigma(t)}^T - P_{\sigma(t)} \tilde{B}_{\sigma(t)} R^{-1} \tilde{B}_{\sigma(t)}^T] P_{\sigma(t)} B.$$

Therefore, one has $P_{\sigma(t)}B = 0, \forall t$.

(ii) Multiplying matrix B to the right of (53), we can get

$$K_1 B = R^{-1}(\tilde{B}^T_{\sigma(t)} P_{\sigma(t)} + H^T)B.$$

Since $P_{\sigma(t)}B = 0$ and $H^T B = R$, then $K_1 B = I, \forall t$.

(iii) According to linear transformation (9), we obtain that

$$K_1 x(t) = K_1[x_1(t) + Bu_1(t)] = K_1 x_1(t) + K_1 Bu_1(t). \tag{59}$$

In the above formula, substituting (52) and (55), one has

$$\begin{aligned} K_1 x(t) &= K_1 x_1(t) + K_1 Bu_1(t) \\ &= K_1 x_1(t) + K_1 B[-K_1 x_1(t)] \\ &= K_1 x_1(t) - K_1 BK_1 x_1(t) \\ &= K_1 x_1(t) - IK_1 x_1(t) \\ &= 0 \end{aligned} \tag{60}$$

Therefore, (56) is immediately proved.

Using the solution of Problem 2 to solve Problem 1 needs to satisfy the following conditions. According to the solution of Problem 2, by (9), we can obtain the following boundary constraints:

$$x(t_0) = x_1(t_0) + Bu_1(t_0), \tag{61}$$

where $x(t_0)$ is any given initial state in problem 1. Therefore, the initial state $x_1(t_0)$ in problem 2 dependent on the initial state $x(t_0)$, and the boundary constraint (61) must be satisfied.

When $t = t_0$, by Theorem 2, we have

$$u_1(t_0) = -K_1 x_1(t_0).$$

Then, the boundary condition (61) becomes

$$x(t_0) = x_1(t_0) - BK_1 x_1(t_0). \tag{62}$$

Now, the following theorem will give the necessary and sufficient conditions for (62) established.

Theorem 3: The necessary and sufficient condition for the establishment of the boundary constraint (62) is

$$K_1 x(t) = 0, \forall t.$$

Proof: *(Necessity)* Multiplying the matrix K_1 to the left side of (62), we can get

$$K_1 x(t_0) = K_1 x_1(t_0) - K_1 BK_1 x_1(t_0).$$

From (55), we have that

$$K_1 B = I.$$

Therefore, the necessity holds for $K_1 x(t_0) = 0$.

(Sufficiency) Let $K_1 x(t_0) = 0$ and choose $x(t_0) = x_1(t_0)$. Then, we obtain that

$$K_1 x(t_0) = K_1 x_1(t_0) = 0,$$

in which

$$u_1(t_0) = -K_1 x_1(t_0) = 0.$$

Therefore, one has

$$x(t_0) = x_1(t_0) - BK_1x_1(t_0).$$

In order to convert the solution of Problem 2 back to the solution of Problem 1, we require that the relationship (9) is established when $t = t_0$, and that (9) can also be established when $t > t_0$. The following theorem will give an explanation.

Theorem 4: If

$$x(t) = x_1(t) + Bu_1(t) \tag{63}$$

holds at $t = t_0$, then (63) still holds for all $t > t_0$.

Proof: For all $t > t_0$, taking the derivative of (63) with respect to time t, we have

$$\begin{aligned}\frac{d}{dt}[x(t) - x_1(t) - Bu_1(t)] &= \dot{x}(t) - \dot{x}_1(t) - B\dot{u}_1(t)\\ &= A_{\sigma(t)}x(t) + Bu(t) - A_{\sigma(t)}x_1(t) - \tilde{B}_{\sigma(t)}u_1(t) - Bu(t)\\ &= A_{\sigma(t)}x(t) - A_{\sigma(t)}x_1(t) - A_{\sigma(t)}Bu_1(t)\\ &= A_{\sigma(t)}[x(t) - x_1(t) - Bu_1(t)].\end{aligned} \tag{64}$$

Obviously (64) is a homogeneous equation of state. Let the state transition matrix is $\Phi(t,t_0)$, then the solution of (64) is

$$[x(t) - x_1(t) - Bu_1(t)] = \Phi(t,t_0)[x(t_0) - x_1(t_0) - Bu_1(t_0)]. \tag{65}$$

In (65), $\Phi(t,t_0)$ is a non-singular matrix, and by (61), we proved

$$x(t) = x_1(t) + Bu_1(t), \forall t.$$

From Theorem 3 and Theorem 4, the solution of the Problem 2 must be transformed into the solution of the Problem 1. Therefore, from (10), we obtain that the optimal control input of Problem 1 is

$$\begin{aligned}u(t) &= \dot{u}_1(t)\\ &= -K_1\dot{x}_1(t)\\ &= -K_1[A_{\sigma(t)}x_1(t) + \tilde{B}_{\sigma(t)}u_1(t)]\\ &= -K_1[A_{\sigma(t)}x_1(t) + A_{\sigma(t)}Bu_1(t)]\\ &= -K_1[A_{\sigma(t)}(x(t) - Bu_1(t)) + A_{\sigma(t)}Bu_1(t)]\\ &= -K_1[A_{\sigma(t)}x(t) - A_{\sigma(t)}Bu_1(t) + A_{\sigma(t)}Bu_1(t)]\\ &= -K_1A_{\sigma(t)}x(t).\end{aligned} \tag{66}$$

Then, the closed-loop switched system becomes

$$\dot{x}(t) = [A_{\sigma(t)} - BK_1A_{\sigma(t)}]x(t). \tag{67}$$

The switching condition of switched system (1) that minimizes the cost function (2) is

$$\begin{aligned}\sigma(t) &= \arg\min_{i\in M}(P_ix_1(t))^T[\tilde{A}_ix_1(t) - \frac{1}{2}\bar{B}_{ii}P_ix_1(t)]\\ &= \arg\min_{i\in M} x_1^T(t)P_i(\tilde{A}_i - \frac{1}{2}\bar{B}_{ii}P_i)x_1(t)\\ &= \arg\min_{i\in M}[x(t) - Bu_1(t)]^T P_i(\tilde{A}_i - \frac{1}{2}\bar{B}_{ii}P_i)[x(t) - Bu_1(t)],\end{aligned} \tag{68}$$

where each $P_i, \forall i \in M$, satisfies the algebra Riccati equation (49).

4. An Illustrative Example

Consider a switched linear system (1) with the following subsystems:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bu(t), w = [1,0,0] \\ A_2 x(t) + Bu(t), w = [0,1,0], \\ A_3 x(t) + Bu(t), w = [0,0,1] \end{cases}$$

with

$$A_1 = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix}, A_1 = \begin{bmatrix} -3 & 2 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

The cost function is defined as

$$J = \frac{1}{2}\int_{t_0}^{t_f} x^T(t)Qx(t)dt, \tag{69}$$

where $Q = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, initial time $t_0 = 0$, and final time $t_f = 100$.

Our purpose is minimizing the cost function (69) by designing switching signal and controllers for each subsystems.

It should be noted that each subsystem $(A_i, B), \forall i \in M = \{1,2,3\}$, is controllable.

By our proposed method in previous section, we define the linear transformation

$$\begin{aligned} x_1(t) &= x(t) - Bu_1(t), \\ \dot{u}_1(t) &= u(t). \end{aligned} \tag{70}$$

Then, we obtain the switched linear system (19) with the following subsystems:

$$\dot{x}_1(t) = \begin{cases} \tilde{A}_1 x_1(t) + \tilde{B}_1 u_1(t), w = [1,0,0] \\ \tilde{A}_2 x_1(t) + \tilde{B}_2 u_2(t), w = [0,1,0], \\ \tilde{A}_3 x_1(t) + \tilde{B}_3 u_3(t), w = [0,0,1] \end{cases}$$

Where

$$\tilde{A}_1 = \begin{bmatrix} -1/21 & 2/63 \\ -4/7 & 8/21 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} -50/21 & 100/63 \\ -8/7 & 16/21 \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} -3 & 2 \\ -12/7 & 8/7 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} -2 \\ -6 \end{bmatrix}, \tilde{B}_3 = \begin{bmatrix} 0 \\ -9 \end{bmatrix}. \tag{71}$$

It is known that each subsystem $(\tilde{A}_i, \tilde{B}_i)$ is controllable. Thus, the cost function (69) can be converted to the following:

$$J = \frac{1}{2}\int_{t_0}^{t_f} [x_1^T(t)Q_1 x_1(t) + u_2^T(t)Ru_2(t)]dt.$$

By (14) and (16), we obtain $R = 63$ and $Q_1 = \begin{bmatrix} 5/7 & -10/21 \\ -10/21 & 20/63 \end{bmatrix}$. By solving (49), we get

$$P_1 = \begin{bmatrix} 1400/947 & -1161/1178 \\ -1161/1178 & 387/589 \end{bmatrix}, P_2 = \begin{bmatrix} 700/3187 & -1482/10121 \\ -1482/10121 & 1070/10961 \end{bmatrix}, \\ P_{31} = \begin{bmatrix} 329/1760 & -329/2640 \\ -329/2640 & 329/3960 \end{bmatrix}. \tag{72}$$

Therefore, we design the switching signal of the switched system:

$$\sigma(t) = \arg\min_{i \in M} [x(t) - Bu_1(t)]^T P_i(\tilde{A}_i - \frac{1}{2}\bar{B}_{ii}P_i)[x(t) - Bu_1(t)], \quad (73)$$

and the controllers for each subsystems:

$$u(t) = -R^{-1}(\tilde{B}^T_{\sigma(t)}P_{\sigma(t)} + H^T)A_{\sigma(t)}x(t).$$

Choose the initial state $x(0) = [2,2]^T$ and the costate vector $\lambda(0) = [478/485, -387/589]^T$. The state trajectories under switched LQR are shown in Figure 1. The optimal switching control and input control are shown in Figure 3 and Figure 2, respectively.

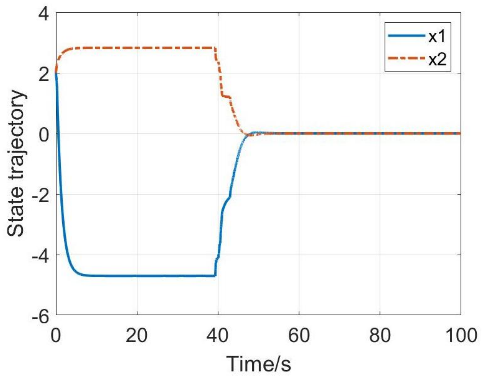

Figure 1. The state trajectories of switched system

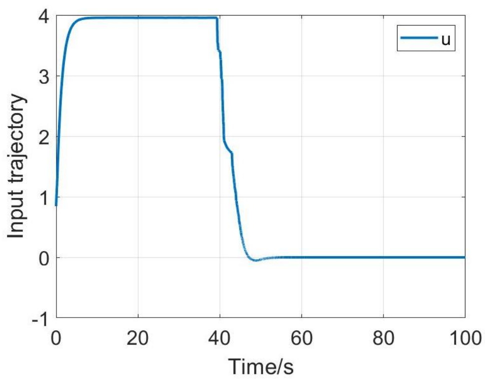

Figure 2. The input trajectories of switched system

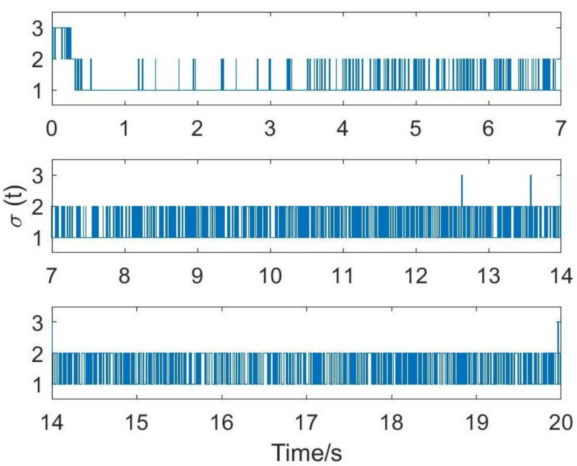

Figure 3. Switching signal $\sigma(t)$

5. Conclusions

This paper has dealt with singular optimal linear quadratic regulator of switched systems, where the controlled variable is comprised of the switch signal as well as the control input. We have investigated on a finite time horizon and solved them by the linear transformation and embedding transformation method. The Hessian matrices of the Hamilton functions have been proven to be negative semi-definite, which leads to bang-bang type solutions of the optimization problems. The switching condition is obtained by solving the Riccati equation, and then the optimal switching instants and optimal mode selection are obtained. Finally, a numerical example has illustrated the efficacy of the proposed method.

References

Bijl Hildo & Schon Thomas B. (2019). Optimal controller/observer gains of discounted-cost LQG systems. *Automatica, 101*, 471-474.

Chen Weisheng, Wen Changyun & Wu Jian. (2018). Global Exponential/Finite-Time Stability of Nonlinear Adaptive Switching Systems with Applications in Controlling Systems with Unknown Control Direction. *IEEE Transactions on Automatic Control, 63*(8), 2738-2744.

Duarte J. Antunes & W.P.M.H. Heemels. (2017). Linear quadratic regulation of switched systems using informed policies. *IEEE Transactions on Automatic Control, 62*, 2675-2688.

Fu Jun, Ma Ruicheng & Chai Tianyou. (2015). Global Finite-Time Stabilization of a Class of Switched Nonlinear Systems with the Powers of Positive Odd Rational Numbers. *Automatica, 54*, 360-373.

Lee Ti-Chung, Tan Ying & Mareels Iven M. Y. (2017). Analyzing the Stability of Switched Systems Using Common Zeroing-Output Systems. *IEEE Transactions on Automatic*

Control, 62, 5138-5153.

Li Li Li, Zhao, Jun and Dimirovski, Georgi M. (2013). Multiple Lyapunov functions approach to observer-based H2 control for switched systems. *International Journal of Systems Science, 44*(4-6), 812-819.

Liu Xiaomeng, Lin Hai & Chen Ben M. (2013). Structural Controllability of Switched Linear Systems. *Automatica, 49*(12), 3531-3537.

Liu Xiaomeng, Lin Hai & Chen, Ben M. (2013). Structural Controllability of Switched Linear Systems. *Automatica, 49*(12), 3531-3537.

Lu Junjie, She Zhikun, Feng Weijie & Ge Shuzhi Sam. (2018). Stabilizability of Time-Varying Switched Systems Based on Piecewise Continuous Scalar Functions. *IEEE Transactions on Automatic Control, 64*, 2637-2644.

Luus Rein, Chen Yang Quan. (2004). Optimal switching control via direct search optimization. *Asian Journal of Control, 6*(2), 302-306.

Ma Ruicheng & An Shuang. (2019). Minimum Dwell Time for Global Exponential Stability of a Class of Switched Positive Nonlinear Systems. *IEEE/CAA Journal of Automatica Sinica, 000*(002), 471-477.

Ma Ruicheng, Chen Qi, Zhao Shengzhi & Fu Jun. (2021). Dwell-Time-Based Exponential Stabilization of Switched Positive Systems with Actuator Saturation. *IEEE Transactions on Systems, Man, and Cybernetics: Systems, 51*, 7685-7691.

Ma Ruicheng, Liu Yan, Zhao Shengzhi, Wang Min, Zong Guangdeng. (2015). Global stabilization design for switched power integrator triangular systems with different powers. *Nonlinear Analysis: Hybrid Systems, 15*(2), 74-85.

Ma Ruicheng, Ma Mingjun, Li Jinghan, Fu Jun and Wu Caiyun. (2019). Standard H2 performance of switched delay systems under minimum dwell time switching. *Journal of the Franklin Institute, 356*(6), 3443-3456.

Niu Teng, Zhai Jingang, Yin Hongchao and Feng Enmin. (2018). Optimal control of nonlinear switched system in an uncoupled microbial fed-batch fermentation process. *Journal of the Franklin Institute, 355*(14), 6169-6190.

Riedinger and Pierre. (2014). A switched LQ regulator design in continuous time. *IEEE Transactions on Automatic Control, 59*(5), 1322-1328.

Seatzu Carla, Corona Daniele, Giua Alessandro & Bemporad, Alberto. (2006). Optimal control of continuous-time switched affine systems. *IEEE Transactions on Automatic Control, 51*(5), 726-741.

Seatzu Carla, Corona Daniele, Giua Alessandro and Bemporad Alberto. (2006). Optimal control of continuous-time switched affine systems. *IEEE Transactions on Automatic Control, 51*(5), 726-741.

Sorin C. Bengea and Raymond A. DeCarlo. (2005). Optimal control of switching systems. *Automatica, 41*(1), 11-27.

Tanwani Aneel, Shim Hyungbo & Liberzon Daniel. (2013). Observability for Switched Linear Systems: Characterization and Observer Design. *IEEE Transactions on Automatic Control, 58*(4), 891-904.

Wang Min and Zhao Jun. (2010). Quadratic stabilization of a class of switched nonlinear systems via single Lyapunov function. *Nonlinear Analysis: Hybrid Systems, 4*(1), 44-53.

Wu Guangyu, Sun Jian & Chen, Jie. (2019). Optimal linear quadratic regulator of switched systems. *IEEE Transactions on Automatic Control, 64*(7), 2898-2904.

Wu Weiping, Gao Jianjun, Lu Jun Guo & Li, Xun. (2020). On continuous-time constrained stochastic linear quadratic control. *Automatica, 114*, 108809.

Xu Wei, Feng Zhi Guo, Lin Gui Hua, Yiu Ka Fai Cedric & Yu Liying. (2020). Optimal Switching of Switched Systems with Time Delay in Discrete Time. *Automatica, 112*, 108696.

Xu Wei, Feng Zhi Guo, Peng Jian Wen, Yiu Ka Fai Cedric. (2017). Optimal switching for linear quadratic problem of switched systems in discrete time. *Automatica, 78*, 185-193.

Xu Xuping, Antsaklis Panos J. (2004). Optimal control of switched systems based on parameterization of the switching instants. *IEEE Transactions on Automatic Control, 49*(1), 2-16.

Xu, Wei, Feng, Zhi Guo, Lin, Gui-Hua, Yiu, Ka Fai Cedric and Yu, Liying. (2020). Optimal switching of switched systems with time delay in discrete time. *Automatica, 112*, 108696.