PIONEER
ACADEMIC PUBLISHING LIMITED

Journal of Progress in Engineering and Physical Science
ISSN 2709-4006
www.pioneerpublisher.com/jpeps
Volume 2 Number 2 June 2023

The Relativistic Equations of Motion for a Charged Particle in the Magnetic Field of an Infinite Current-Carrying Wire

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doi:10.56397/JPEPS.2023.06.01

Abstract

We show how to produce the closed form solution for the motion of a charged particle in the magnetic field of an infinitely long, current-carrying wire in the relativistic range, thus extending the results produced recently (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019). We outline the areas where the two solutions are similar aa well as where the two solutions are slightly different. We also extend the work in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) to the more realistic case where the current generates both a magnetic and an electric field, as is the case in real life. We plan to extend the approach in the future to more complicated cases as the one of the finite length wire. The solution is of great interest for the design of particle accelerators, hence it is interesting for both theoretical physicists and engineers alike.

Keywords: closed solutions, trajectory of a charged particle moving at relativistic speeds in the magnetic field of an infinite current carrying wire, Biot-Savart law

1. Introduction

Extensive treatment of the trajectories of charged particles moving at non-relativistic speeds in a magnetic field abound in scientific literature (Cook, M. D., 1970; Hurley, J., 1961; Hertweck, F., 1959; Muller, M., & Dietrich, K., 1995; Yafaev, D., 2003; Essen, H., & Nordmark, A., 2016; Prentice, A., Fatuzzo, M., & Toepker, T., 2015; Brizard, A. J., 2017; Seymour, P. W., 1959; Stetson, R. F., Lamborn, B. N. A., & Lafferty, D. L., 1963; Huggins, E. R., & Lelek, J. J., 1979; McGuire, G. C., 1979; Öztürk, M. K., 2012; Aguirre, A. J., Luque, B. A., & Peralta-Salas, D., 2010). An exhaustive relativistic treatment for the case of arbitrary stationary electromagnetic fields can be found in (Sfarti, A., 2011). Treatments for constant magnetic fields are quite common in literature, on the other hand, if the magnetic field is not constant this will make the problem significantly more difficult to solve. A solution for the non-constant magnetic field of an infinitely long wire has only been produced recently (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) but only for the non-relativistic regime. In the current paper we are providing the most general, fully relativistic solution for the case of a charged particle moving in the variable magnetic field created by an infinitely long, current-carrying wire. Such a field is obviously non uniform, thus making the problem somewhat challenging. Yet, the solution turned out to be quite elegant. The idea behind this research is that different wire and loop configurations can be used in order to accelerate charged particles up to relativistic velocities, thus creating methods for "steering" the particles in practical applications of particle accelerators.

2. The Solution

In what follows we use the exact notation used in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019). In the cylindrical coordinate system (s,φ,z), an infinitely long wire is located on the z-axis. The magnetic field produced by the current of intensity I is (see figure 1):

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}} \tag{1.1}$$

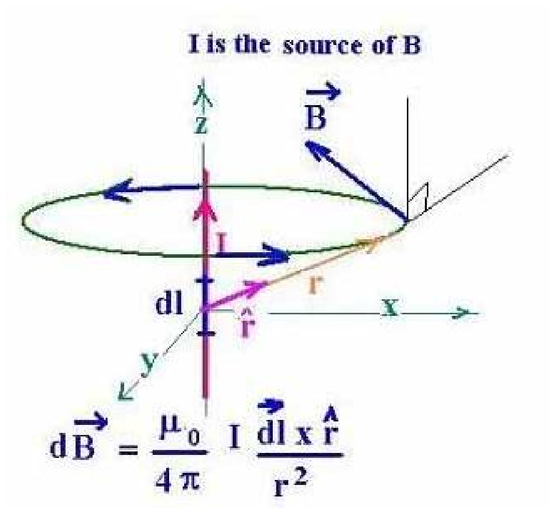

Figure 1.

The Lorentz force exerted on a particle of charge q and mass m entering the magnetic field at initial velocity $\mathbf{v_0}$ and subsequently moving with the instantaneous velocity $\mathbf{v}$ is:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1.2}$$

Because the Lorentz force is perpendicular on the particle velocity, the speed of the particle is constant $v = v_0$ (Sfarti, A., 2011) while the velocity is variable. On the other hand, the relativistic force exerted on the particle of mass m moving with instantaneous velocity $\mathbf{v}$ is, according to (Sfarti, A., 2011):

$$\mathbf{F} = m\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{m}{\sqrt{1-\frac{v_0^2}{c^2}}}\frac{d\mathbf{v}}{dt} = m\gamma_0\frac{d\mathbf{v}}{dt} \tag{1.3}$$

From (1,1), (1.2) and (1.3) we obtain:

$$\frac{dv_s}{dt} = -\frac{q\mu_0 I}{2\pi m\gamma_0 s}v_z = -\frac{v_L}{\gamma_0 s}v_z \tag{1.4a}$$

$$\frac{dv_\varphi}{dt} = 0 \tag{1.4b}$$

$$\frac{dv_z}{dt} = \frac{v_L}{\gamma_0 s}v_s \tag{1.4c}$$

where $v_L = \frac{q\mu_0 I}{2\pi m}$ is the Larmor speed. The above equations correspond exactly to equations (3a-3c) from reference (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) with the right hand term scaled by the constant γ_0. Therefore, the general case, valid at any speeds, including the relativistic ones, is solved by reducing the problem to one that has already been solved by simply scaling the Larmor speed v_L by a constant. The net effect results into scaling variables u_ρ, u_ζ (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) by the same constant, γ_0.

As a consequence, equations (4a-4d) in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) are unchanged. As a final consequence, the closed solution for the equations of motion expressed by (7), (8) and (12) in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) also holds. Using the same notations as in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019), the equations of motion are:

$$\cos\theta = \cos\theta_0 - \frac{\ln\rho}{u_{0\perp}}$$

$$\rho = \frac{e^{u_{0\perp}\cos\theta}}{e^{u_{0\perp}\cos\theta_0}} \tag{1.5}$$

$$\zeta = \zeta_0 + \frac{u_{0\perp}}{e^{u_{0\perp}\cos\theta_0}} \int_{\theta_0}^{\theta} e^{u_{0\perp}\cos\theta} \cos\theta d\theta$$

The period of the motion is also identical with the one calculated in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) for the non-relativistic case:

$$T = \frac{2\pi I_0 u_{0\perp}}{e^{u_{0\perp}\cos\theta_0}} \tag{1.6}$$

In (1.6) I_0 is the modified Bessel function of zeroth order.

Naturally, not all formulas from (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019) apply in the relativistic regime. For example, the relativistic kinetic energy has a much simpler form than the corresponding formula from reference (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019):

$$K = (\gamma_0 - 1)mc^2 \tag{1.7}$$

The reason for the above is that no approximations are necessary when calculating the kinetic energy in the relativistic regime.

3. A More Comprehensive Case

The case treated above (and in (Asadi-Zeydabadi, M., & Zaidins, C. S., 2019)) ignores the fact that the current in the infinite wire creates not only a magnetic field but also an electric one:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}} \tag{2.1}$$

In (2.1) λ represents the charge density. Therefore, the Lorentz force takes the general form:

$$\mathbf{F}_L = q(\mathbf{v}\times\mathbf{B} + \mathbf{E}) = q((E_s - B_\varphi v_\zeta)\hat{\mathbf{s}} + v_s B_\varphi \hat{\boldsymbol{\zeta}}) \tag{2.2}$$

The relativistic force becomes much more complex as well:

$$\mathbf{F} = m\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m\frac{(1-\frac{v^2}{c^2})\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}v}{c^2}\frac{dv}{dt}}{(1-\frac{v^2}{c^2})^{3/2}} = m\frac{(1-\frac{v^2}{c^2})\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}}{2c^2}\frac{dv^2}{dt}}{(1-\frac{v^2}{c^2})^{3/2}} =$$
$$= m\frac{(1-\frac{v^2}{c^2})\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}}{c^2}\sum v_s\frac{dv_s}{dt}}{(1-\frac{v^2}{c^2})^{3/2}} \tag{2.3}$$

In (2.3) the speed is no longer constant:

$$v^2 = v_s^2 + v_\varphi^2 + v_\zeta^2$$
$$\mathbf{v} = v_s\hat{\mathbf{s}} + v_\varphi\hat{\boldsymbol{\varphi}} + v_\zeta\hat{\boldsymbol{\zeta}} \tag{2.4}$$

Therefore, the equations of motion obtained from:

$$\mathbf{F} = \mathbf{F_L} \tag{2.5}$$

no longer have a closed form. We cannot apply the change of coordinate system described in (Sfarti, A., 2011) because $\mathbf{B}, \mathbf{E}$ are variable. The presence of the factors in v^2 precludes us from getting the simple expressions from the previous section. The only available avenue left for integrating the equations of motion is the numerical one.

4. Conclusion and Future Work

We have solved the general, relativistic case of a charged particle moving in the magnetic field produced by an infinitely long wire by reducing the problem to one that has already been solved. We have shown that the equations of motion and the period of motion have the same form as the equations for the non-relativistic case, modulo the fact that the Larmor speed v_L needed to be scaled to v_L/γ_0. On the other hand, the kinetic energy formula K has a much simpler form. We have also added the treatment for the real life case where the current creates an electric field in addition to the magnetic one. The solution is of great interest for the design of particle accelerators, hence it is interesting for both theoretical physicists and engineers alike. We plan to extend the current work to the more challenging case of a wire of finite length. The case is interesting for practical reasons and it is challenging because of the more complicated form of the Biot – Savart law.

PACS: 03.30.+p, 52.20.Dq, 52.70.Nc

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